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QUASI-INVARIANT DOMAIN CONSTANTS*

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ABSTRACT

Five domain constants are studied in our paper, all related to the hyperbolic geometry in hyperbolic plane regions which are uniformly perfect (in Pommerenke's terminology). Relations among these domain constants are obtained, from which bounds are derived for the variance ratio of each constant under conformal mappings of the regions, and we also show that each constant may be used to characterize uniformly perfect regions.

1. Introduction

We start by recalling some facts about the hyperbolic metric. The hyperbolic metric on the unit disk $\mathbb{D}$ is given by

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{|dz|}{1-|z|^2}.$$

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It is normalized to have Gaussian curvature -4. For a hyperbolic region Ω in the complex plane $\mathbb{C}$, that is, $\mathbb{C}\setminus\Omega$ contains at least two points, let $\lambda_\Omega(z)|dz|$ denote the hyperbolic metric on Ω with curvature -4. It is obtained from

$$\lambda_\Omega(f(w))|f'(w)| = \lambda_{\mathbb{D}}(w) = \frac{1}{1-w\bar{w}}, \tag{1}$$

where $f : \mathbb{D} \to \Omega$ is any holomorphic universal covering projection. Let d_Ω denote the hyperbolic distance function on Ω determined from the hyperbolic metric. Note that

$$d_{\mathbb{D}}(a,b) = \operatorname{artanh}\left|\frac{a-b}{1-a\bar{b}}\right|.$$

The pseudohyperbolic distance is given by $\tanh d_\Omega$. Also, the hyperbolic disk with center a and hyperbolic radius r is $D_\Omega(a,r) = \{z \in \Omega : d_\Omega(a,z) < r\}$.

The quasi-hyperbolic metric on a region $\Omega \neq \mathbb{C}$ is defined by $|dz|/\delta_\Omega(z)$, where $\delta_\Omega(z) = \operatorname{dist}(z,\partial\Omega)$ is the euclidean distance from z to $\partial\Omega$. By Schwarz' Lemma the ratio $\lambda_\Omega(z)\delta_\Omega(z)$ of the hyperbolic and quasi-hyperbolic metrics on a hyperbolic region Ω is always bounded above by 1 [5, p. 45]. Thus, as long as the domain constant

$$c(\Omega) = \inf\{\lambda_\Omega(z)\delta_\Omega(z) : z \in \Omega\}$$

is positive, the hyperbolic and quasi-hyperbolic metrics are comparable. A region Ω for which $c(\Omega) > 0$ is called **uniformly perfect** following Pommerenke [11]. The punctured disk $\mathbb{D}^* = \mathbb{D}\setminus\{0\}$, or more generally, any hyperbolic region containing an isolated finite boundary point, is not uniformly perfect. Osgood [9, Cor. 1] showed that if Ω and Δ are conformally equivalent hyperbolic regions, then

$$\frac{1}{B} \le \frac{c(\Omega)}{c(\Delta)} \le B$$

for some positive constant $B \le 6$. Therefore, the domain constant $c(\Omega)$ is quasi-invariant under conformal mappings and so Ω is uniformly perfect if and only if Δ is. Minda [7] improved this estimate on B to $B \le 4\coth(\pi/2\sqrt{3}) = 5.5583\cdots$. In this paper we improve this bound to

$$2 \le B \le \sqrt{1+3\coth^2\left(\frac{\pi}{4}\right)} = 2.824\cdots.$$

One ingredient in this proof is the fact that $c(\Omega) \le \frac{1}{2}$ for any hyperbolic region Ω with equality if and only if Ω is convex. We conjecture that $B = 2$. For a simply connected domain Ω, $c(\Omega) \geq \frac{1}{4}$ with equality if Ω is a slit plane [5, p. 45]. Thus, for the class of simply connected hyperbolic domains we have $\frac{1}{2} \leq c(\Omega)/c(\Delta) \leq 2$.

We also study four other domain constants defined for hyperbolic domains; three of these are quasi-invariant under conformal mappings while the fourth is a conformal invariant. We show that all four of these constants can be used to characterize uniformly perfect regions. For these three quasi-invariant domain constants we show that the variance ratio under a conformal mapping is the same for simply connected domains as for general hyperbolic domains. This lends support to our conjecture that $B = 2$.

2. Domain Constants

In this section we define four domain constants for any hyperbolic region Ω; one of these constants is a conformal invariant while the other three are quasi-invariant. All of these constants are defined in terms of the hyperbolic geometry of the region and also have simple expressions in terms of any holomorphic covering projection of $\mathbb{D}$ onto Ω.

For any hyperbolic domain Ω set

$$\eta(\Omega) = \frac{1}{2} \sup_{z \in \Omega} \frac{|\nabla \log \lambda_\Omega(z)|}{\lambda_\Omega(z)}.$$

It is straightforward to verify that $\eta(\mathbb{D}) = 1$ and $\eta(\mathbb{D}^*) = \infty$.

The quantity $\eta(\Omega)$ can be expressed conveniently in terms of any holomorphic universal covering projection $f : \mathbb{D} \to \Omega$. For such a covering projection the identity (1) holds. By taking the logarithm of (1) and differentiating, we obtain

$$\frac{\partial \log \lambda_\Omega(z)}{\partial z} f'(w) + \frac{f''(w)}{2f'(w)} = \frac{\bar{w}}{1 - w\bar{w}}$$

for $z = f(w)$. Then by making use of

$$|\nabla \log \lambda_\Omega(z)| = 2 \left| \frac{\partial \log \lambda_\Omega(z)}{\partial z} \right|,$$

we obtain

$$\frac{|\nabla \log \lambda_\Omega(z)|}{\lambda_\Omega(z)} = \left| (1 - |w|^2) \frac{f''(w)}{f'(w)} - 2\bar{w} \right|.$$

Hence,

$$\eta(\Omega) = \frac{1}{2} \sup_{w \in \mathbb{D}} \left| (1 - |w|^2) \frac{f''(w)}{f'(w)} - 2\bar{w} \right|.$$

This shows that the right-hand side is independent of the choice of the covering projection f of $\mathbb{D}$ onto Ω. The preceding identity also reveals than $\eta(\Omega) = \alpha$, where α is the order [10] of the linearly invariant family

$$\mathcal{G} = \left\{ \frac{f(e^{i\theta} \frac{z+a}{1+\bar{a}z}) - f(e^{i\theta} a)}{(1 - |a|^2) e^{i\theta} f'(e^{i\theta} a)} : a \in \mathbb{D} \text{ and } \theta \in \mathbb{R} \right\}.$$

Recall that the collection of all holomorphic universal covering projections of $\mathbb{D}$ onto Ω is given by

$$\mathcal{F} = \{ f(e^{i\theta}(z + a)/(1 + \bar{a}z)) : a \in \mathbb{D} \text{ and } \theta \in \mathbb{R} \}.$$

In fact,

$$\mathcal{F} = \{ f \circ T : T \text{ is a conformal automorphism of } \mathbb{D} \}.$$

Note that $\mathcal{G}$ consists of the coverings in $\mathcal{F}$ which have been normalized to vanish at the origin and have derivative one at the origin. If $f : \mathbb{D} \to \Omega$ is a covering, $T(z) = (z + a)/(1 + \bar{a}z)$ and $g = f \circ T$, then

$$\frac{g''(0)}{g'(0)} = (1 - |a|^2) \frac{f''(a)}{f'(a)} - 2\bar{a}.$$

Consequently, for any covering $f : \mathbb{D} \to \Omega$

$$\eta(\Omega) = \sup \left\{ \left| \frac{g''(0)}{2g'(0)} \right| : g = f \circ T \text{ and } T \text{ is any conformal automorphism of } \mathbb{D} \right\}.$$

Note that $\eta(\Omega) = 1$ for any convex region [10] and $\eta(\Omega) = 2$ for a slit plane.

The quantity $\eta(\Omega)$ also has geometric significance. If $\gamma : z = z(t), a < t < b$, is a C^2-path in Ω with nonvanishing tangent vector $z'(t)$, then the hyperbolic curvature of γ at the point $z = z(t)$ is given by

$$\kappa_\Omega(z, \gamma) = \frac{\kappa_e(z, \gamma)}{\lambda_\Omega(z)} + 2 \operatorname{Im} \left\{ \frac{\frac{\partial \log \lambda_\Omega(z)}{\partial z}}{\lambda_\Omega(z)} \frac{z'(t)}{|z'(t)|} \right\},$$

where

$$\kappa_e(z, \gamma) = \operatorname{Im} \left\{ \frac{\overline{z'(t)} z''(t)}{|z'(t)|^3} \right\}$$

is the euclidean curvature of γ at $z = z(t)$. One geometric property of $\eta(\Omega)$ is that $\kappa_\Omega(z,\gamma) \geq 2\eta(\Omega)$ implies $\kappa_e(z,\gamma) \geq 0$.

The next domain constant involves the Schwarzian derivative of a conformal metric. Suppose $\rho(z)|dz|$ is a conformal metric on a region Ω. The Schwarzian derivative of this conformal metric is defined by

$$S_\rho(z) = 2\left[\frac{\partial^2 \log\rho(z)}{\partial z^2} - \left(\frac{\partial \log\rho(z)}{\partial z}\right)^2\right] = -2\rho(z)\frac{\partial^2(\rho(z)^{-1})}{\partial z^2}.$$

Here is one justification for calling this expression the Schwarzian derivative of a conformal metric. If h is holomorphic on Ω and $h'(z) \neq 0$ for $z \in \Omega$, then the pull-back of the euclidean metric via h is the conformal metric $|h'(z)|\,|dz| = \rho(z)|dz|$. Straightforward calculation shows that

$$S_\rho(z) = \frac{h'''(z)}{h'(z)} - \frac{3}{2}\left(\frac{h''(z)}{h'(z)}\right)^2 = S_h(z),$$

the usual Schwarzian derivative S_h of the function h.

The Schwarzain derivative of a conformal metric has significance relative to the rate of change of geodesic curvature. If $\gamma : z = z(t), a < t < b$, is a C^2-path in Ω with nonvanishing tangent vector $z'(t)$, then the geodesic curvature of γ at the point $z = z(t)$ relative to the conformal metric $\rho(z)|dz|$ is given by

$$\kappa_\rho(z,\gamma) = \frac{\kappa_e(z,\gamma)}{\rho(z)} + 2\,\mathrm{Im}\left\{\frac{(\partial\log\rho(z)/\partial z)}{\rho(z)}\frac{z'(t)}{|z'(t)|}\right\}.$$

By differentiating this formula with respect to t, we obtain

$$\rho(z)|z'(t)|\frac{d}{dt}\kappa_\rho(z,\gamma) = \frac{d}{dt}\kappa_e(z,\gamma) + \mathrm{Im}\{S_\rho(z(t))z'(t)^2\},$$

where

$$\frac{d}{dt}\kappa_e(z,\gamma) = \frac{1}{|z'(t)|}\mathrm{Im}\left\{\frac{z'''(t)}{z'(t)} - \frac{3}{2}\left(\frac{z''(t)}{z'(t)}\right)^2\right\} = \frac{1}{|z'(t)|}\mathrm{Im}\{S_z(t)\}$$

is the rate of change of euclidean curvature.

The third domain constant is

$$\beta(\Omega) = \frac{1}{2}\sup\left\{\frac{|S_{\lambda_\Omega}(z)|}{\lambda_\Omega(z)^2} : z \in \Omega\right\}.$$

If $f : \mathbb{D} \to \Omega$ is a covering projection, then from (1) we obtain

$$\frac{|S_f(w)|}{\lambda_{\mathbb{D}}(w)^2} = \frac{|S_{\lambda_\Omega}(f(w))|}{\lambda_\Omega(f(w))^2}.$$

Hence,

$$\beta(\Omega) = \tfrac{1}{2}\sup\{(1-|w|^2)^2|S_f(w)| : w \in \mathbb{D}\}.$$

In particular, $\beta(\Omega) = 0$ if and only if Ω is a disk or half-plane since $S_f = 0$ if and only if f is a Möbius transformation. Also, if $\beta(\Omega) \leq 1$, then Ω is simply connected by Nehari's univalence criterion [8]. Observe that $\beta(\Omega) = 1$ for a strip while $\beta(\Omega) = 3$ for a slit plane.

Next, we consider a conformally invariant domain constant. For $a \in \Omega$ let

$$r(a,\Omega) = \sup\{r : D_\Omega(a,r) \text{ is simply connected}\}$$

and

$$r(\Omega) = \inf\{r(a,\Omega) : a \in \Omega\}.$$

The quantity $r(\Omega)$ is called the **hyperbolic radius of injectivity** for Ω, while $R(\Omega) = \tanh r(\Omega)$ is the **pseudo-hyperbolic radius of injectivity**. Because the hyperbolic metric is conformally invariant, it is clear that the hyperbolic radius of injectivity is also a conformal invariant.

We express $r(\Omega)$ in terms of any holomorphic covering projection $f : \mathbb{D} \to \Omega$. Set

$$\rho(w,f) = \sup\{\rho : f \text{ is injective in } D_{\mathbb{D}}(w,\rho)\}$$

and

$$\rho(f) = \inf\{\rho(w,f) : w \in \mathbb{D}\}.$$

If $\rho(f) > 0$, then f is uniformly locally univalent in the hyperbolic sense; that is, f is univalent in every hyperbolic disk with hyperbolic radius $\rho(f)$. Now, it is always true that $f(D_{\mathbb{D}}(w,\rho)) = D_\Omega(f(w),\rho)$ for any hyperbolic disk, but f need not be injective on $D_{\mathbb{D}}(w,\rho)$. If f is injective on $D_{\mathbb{D}}(w,\rho)$, then $D_\Omega(f(w),\rho)$ is simply connected. This yields $\rho(w,f) \leq r(f(w),\Omega)$. On the other hand, if $D_\Omega(f(w),\rho)$ is simply connected, then $f|D_{\mathbb{D}}(w,\rho)$ must be injective. This can be seen as follows. Suppose w_1 and w_2 belong to $D_{\mathbb{D}}(w,\rho)$ and $f(w_1) = f(w_2)$. The hyperbolic geodesic γ connecting w_1 and w_2 lies in $D_{\mathbb{D}}(w,\rho)$ since this disk is hyperbolically convex. Then $\delta = f\circ\gamma$ is a closed path in the simply connected set

$D_\Omega(f(w),\rho)$, so it is homotopically trivial in $D_\Omega(f(w),\rho)$ and consequently homotopically trivial in Ω. This implies that any lift of δ to $\mathbb{D}$ via f is closed. Hence, γ itself is closed and so $w_1 = w_2$. Thus, $f|D_\mathbb{D}(w,\rho)$ is injective when $D_\Omega(f(w),\rho)$ is simply connected, so that $r(f(w),\Omega) \le \rho(w,f)$. Thus, $r(f(w),\Omega) = \rho(w,f)$, which gives $r(\Omega) = \rho(f)$. It is now clear that $R(\Omega) = 1$ if Ω is simply connected.

Our final domain constant is the hyperbolic radius of convexity. For $a \in \Omega$ set

$$r_c(a,\Omega) = \sup\{r : D_\Omega(a,r) \text{ is convex in the euclidean sense}\}.$$

Then

$$r_c(\Omega) = \inf\{r(a,\Omega) : a \in \Omega\}$$

is called the **hyperbolic radius of convexity** and $R_c(\Omega) = \tanh r_c(\Omega)$ is the **pseudo-hyperbolic radius of convexity**. Clearly, $r_c(\Omega) \le r(\Omega)$. Analogous to the situation for the hyperbolic radius of injectivity, one can show that if $f : \mathbb{D} \to \Omega$ is a convering projection,

$$\rho_c(w,f) = \sup\{\rho : f \mid D_\mathbb{D}(w,\rho) \text{ is injective and } f(D_\mathbb{D}(w,\rho)) \text{ is convex}\},$$

and

$$\rho_c(f) = \inf\{\rho_c(w,f) : w \in \mathbb{D}\},$$

then $r_c(f(w),\Omega) = \rho_c(w,f)$ which gives $r_c(\Omega) = \rho_c(f)$. Evidently, $R_c(\Omega) = 1$ if Ω is convex since a convex univalent function maps every subdisk of $\mathbb{D}$ onto a convex set. Also, $R_c(\mathbb{C} \setminus (-\infty, 0]) = 2 - \sqrt{3}$ because $2 - \sqrt{3}$ is the radius of convexity for a univalent function and the Koebe function is extremal.

3. Inequalities for Domain Constants

We now obtain upper and lower bounds for the domain constants $c(\Omega), R_c(\Omega)$, $\beta(\Omega)$ and $\eta(\Omega)$ in terms of the conformal invariant $R(\Omega)$. These bounds enable us to show that these four domain constants are quasi-invariant under conformal mappings.

THEOREM 1: *If Ω is a hyperbolic region in $\mathbb{C}$, then*

$$(2-\sqrt{3})R(\Omega) \le R_c(\Omega) \le R(\Omega).$$

These bounds are best possible.

Proof: The upper bound is elementary and equality holds for a convex region. Now, we establish the lower bound. Fix $a \in \Omega$ and let $f : \mathbb{D} \to \Omega$ be a covering projection with $f(0) = a$. Then f is univalent in $\{z : |z| < R(\Omega)\}$. It follows that $f(\{z : |z| < (2-\sqrt{3})R(\Omega)\}$ is a convex set [2, p. 44]. Therefore,

$$(2-\sqrt{3})R(\Omega) \le R_c(a, \Omega),$$

so the lower bound holds. Since $R_c(\mathbb{C}\setminus(-\infty, 0]) = 2-\sqrt{3}$ and $R(\mathbb{C}\setminus(-\infty, 0]) = 1$, this lower bound is best possible. ∎

COROLLARY: *If Ω and Δ are conformally equivalent hyperbolic regions, then*

$$2-\sqrt{3} \le \frac{R_c(\Omega)}{R_c(\Delta)} \le 2+\sqrt{3}.$$

These bounds are best possible.

Proof: Due to symmetry, it suffices to show that $R_c(\Omega) \le (2+\sqrt{3})R_c(\Delta)$. Now, since the radius of injectivity is a conformal invariant,

$$R_c(\Omega) \le R(\Omega) = R(\Delta) \le \frac{1}{2-\sqrt{3}} R_c(\Delta) = (2+\sqrt{3})R_c(\Delta).$$

Observe that equality holds if $\Omega = \mathbb{D}$ and $\Delta = \mathbb{C}\setminus(-\infty, 0]$. ∎

THEOREM 2: *If Ω is a hyperbolic region in $\mathbb{C}$, then*

$$\frac{1}{R(\Omega)} \le \eta(\Omega) \le \frac{2}{R(\Omega)}.$$

These bounds are best possible.

Proof: We first establish the lower bound. Since the linearly invariant family $\mathcal{G}$ has order $\alpha = \eta(\Omega)$, each function $g \in \mathcal{G}$ is univalent in $\{w : |w| < 1/\alpha\}$ [10]. This actually implies that each $g \in \mathcal{G}$, and so every $f \in \mathcal{F}$, is univalent in the hyperbolic disk $D_{\mathbb{D}}(w, \operatorname{artanh}(1/\alpha))$ for any $w \in \mathbb{D}$. Therefore, $R(\Omega) \ge 1/\alpha$ and the lower bound is established. Equality holds if $\Omega = \mathbb{D}$. The upper bound is also easy to demonstrate. Suppose $f : \mathbb{D} \to \Omega$ is a covering projection. There is nothing to prove if $R = R(\Omega) = 0$, so we may assume that $R > 0$. Then f is univalent in $\{w : |w| < R\}$, so the function $h(w) = [f(Rw) - f(0)]/Rf'(0)$ is a normalized univalent function in $\mathbb{D}$. Therefore, $|h''(0)|/2 \le 2$, or

$$\frac{|f''(0)|}{2|f'(0)|} \le \frac{2}{R}.$$

The same reasoning applies to each function $f \circ T \in \mathcal{F}$, when T is any conformal automorphism of $\mathbb{D}$. Consequently, $\eta(\Omega) = \alpha \le 2/R$. Note that equality holds for $\Omega = \mathbb{C}\setminus(-\infty, 0]$. ∎

COROLLARY: *If Ω and Δ are conformally equivalent hyperbolic regions, then*

$$\frac{1}{2} \leq \frac{\eta(\Omega)}{\eta(\Delta)} \leq 2.$$

These bounds are best possible.

Proof: Because of symmetry we need only show that $\eta(\Omega) \leq 2\eta(\Delta)$. Since the radius of injectivity is a conformal invariant,

$$\eta(\Omega) \leq \frac{2}{R(\Omega)} = \frac{2}{R(\Delta)} \leq 2\eta(\Delta).$$

Since $\eta(\mathbb{D}) = 1$ and $\eta(\mathbb{C} \setminus (-\infty, 0]) = 2$, this inequality is best possible. ∎

THEOREM 3: *If Ω is a hyperbolic region in $\mathbb{C}$, then*

$$\beta(\Omega) \leq \frac{3}{R(\Omega)^2}.$$

If $\beta(\Omega) \geq 1$, then

$$\frac{1}{R(\Omega)^2} \leq \beta(\Omega).$$

These bounds are best possible.

Proof: We begin by establishing the upper bound. We may assume that $R = R(\Omega) > 0$. Fix $b \in \mathbb{D}$. Define $T(w) = (Rw + b)/(1 + \bar{b}Rw)$. Then T is a conformal mapping of $\mathbb{D}$ onto $D_{\mathbb{D}}(b, \rho)$ where $\rho = \operatorname{artanh} R$. Thus, $g = f \circ T$ is univalent in $\mathbb{D}$, so the Kraus–Nehari inequality gives $(1 - |w|^2)^2 |S_g(w)| \leq 6$. From $S_g = (S_f \circ T)(T')^2$, we obtain

$$6 \geq |S_g(0)| = |S_f(T(b))| \, |T'(b)|^2 = |S_f(b)|(1 - |b|^2)^2 R^2.$$

This yields $\beta(\Omega) \leq 3/R(\Omega)^2$. Equality holds for $\Omega = \mathbb{C} \setminus (-\infty, 0]$.

The lower bound is established in a similar manner. We assume that $1 \leq \beta(\Omega) < \infty$. Set $R = 1/\sqrt{\beta(\Omega)}$. Fix $b \in \mathbb{D}$ and define T as in the first portion of the proof. We will show that $g = f \circ T$ is univalent in $\mathbb{D}$. This implies that f is univalent in the disk $D_{\mathbb{D}}(b, \operatorname{artanh} R)$ so $\rho(b, f) \geq \operatorname{artanh} R$ and

$$R(\Omega) = \tanh \rho(f) \geq 1/\sqrt{\beta(\Omega)},$$

which is the desired result. Equality holds for $\Omega = \{z : |\operatorname{Im} z| < \pi/2\}$. Now,

$$(1-|w|^2)^2|S_g(w)| = (1-|T(w)|^2)^2|S_f(T(w))|\left(\frac{(1-|w|^2)|T'(w)|}{(1-|T(w)|^2)}\right)^2$$
$$\leq 2\beta(\Omega)\left(\frac{(1-|w|^2)|T'(w)|}{(1-|T(w)|^2)}\right)^2.$$

Direct calculation gives

$$\frac{(1-|w|^2)|T'(w)|}{(1-|T(w)|^2)} = \frac{(1-|w|^2)R}{1-R^2|w|^2} \leq R,$$

so that

$$(1-|w|^2)^2|S_g(w)| \leq 2\beta(\Omega)R^2 = 2.$$

Nehari's univalence criterion [8] implies that g is univalent in $\mathbb{D}$. ∎

COROLLARY: *If Ω and Δ are conformally equivalent hyperbolic regions and $\beta(\Omega), \beta(\Delta)$ are both at least 1, then*

$$\frac{1}{3} \leq \frac{\beta(\Omega)}{\beta(\Delta)} \leq 3.$$

In particular, the above inequality holds whenever Ω and Δ are non-simply connected hyperbolic regions. These bounds are best possible.

Proof: We need only show that $\beta(\Omega) \leq 3\beta(\Delta)$. Because the radius of injectivity is conformally invariant,

$$\beta(\Omega) \leq \frac{3}{R(\Omega)^2} = \frac{3}{R(\Delta)^2} \leq 3\beta(\Delta).$$

Equality holds if $\Omega = \mathbb{C} \setminus (-\infty, 0]$ and $\Delta = \{z : |\operatorname{Im} z| < \pi/2\}$.

The following theorem is due to Hilditch [4]. The authors rediscovered this result and found several different proofs. We present Hilditch's proof since it is the most elementary. ∎

THEOREM 4: *If Ω is a hyperbolic region in $\mathbb{C}$, then $c(\Omega) \leq 1/2$ with equality if and only if Ω is convex.*

Proof: Recall that Ω is convex if and only if $c(\Omega) \geq 1/2$ ([4], [6]). Therefore, it suffices to show that $c(\Omega) \leq 1/2$ for any hyperbolic region Ω. Fix $a \in \Omega$. Select

$c \in \partial\Omega$ with $|a-c| = d = \delta_\Omega(a)$. Then the disk $\Delta = \{z : |z-a| < d\}$ is contained in Ω, so the monotonicity property of the hyperbolic metric gives

$$\lambda_\Omega(z) \le \lambda_\Delta(z) = \frac{d}{d^2 - |z-a|^2}$$

for $z \in \Delta$. If z belongs to the half-open segment $[a, c)$, then $\delta_\Omega(z) = d - |z-a|$. Hence, for $z \in [a, c)$

$$c(\Omega) \le \lambda_\Omega(z)\delta_\Omega(z) = \frac{d}{d + |z-a|}.$$

If we let $z \to c$ along the segment $[a, c)$, then we obtain $c(\Omega) \le 1/2$. ∎

THEOREM 5: *If Ω is a hyperbolic region in $\mathbb{C}$, then*

$$\frac{R(\Omega)}{2\sqrt{3 + R(\Omega)^2}} \le c(\Omega) \le \min\left\{\frac{1}{2}, \frac{2}{\pi}\operatorname{artanh} R(\Omega)\right\}.$$

Proof: We begin by establishing the lower bound. From Theorem 3 we have

$$\beta(\Omega) \le \frac{3}{R(\Omega)^2}.$$

Since $\eta(\Omega)^2 - 1 \le \beta(\Omega)$ [10], we obtain

$$\eta(\Omega) \le \frac{\sqrt{3 + R(\Omega)^2}}{R(\Omega)}.$$

Because $1/2\eta(\Omega) \le c(\Omega)$ [9], we get the lower bound. Equality holds for a slit plane.

The proof of the upper bound is similar. First, ([1], [7], [12]) if $\beta(\Omega) > 1$, then

$$\tanh\frac{\pi}{2\sqrt{\beta(\Omega) - 1}} \le R(\Omega).$$

Harmelin [3] proved that $\beta(\Omega) \le 1 + \eta(\Omega)^2$, so that

$$\tanh\frac{\pi}{2\eta(\Omega)} \le R(\Omega).$$

Since $c(\Omega) \le 1/\eta(\Omega)$ [9], we get

$$\tanh\frac{\pi c(\Omega)}{2} \le R(\Omega).$$

This establishes the upper bound when $\beta(\Omega) > 1$. If $\beta(\Omega) \le 1$, then Ω is simply connected [8]. In this case the upper bound is trivial since $R(\Omega) = 1$ implies

$$(2/\pi)\operatorname{artanh} R(\Omega) = \infty > 1/2 \ge c(\Omega).$$

COROLLARY: *If Ω and Δ are conformally equivalent hyperbolic regions, then*

$$\frac{1}{\sqrt{1+3\coth^2(\frac{\pi}{4})}} \le \frac{c(\Omega)}{c(\Delta)} \le \sqrt{1+3\coth^2\left(\frac{\pi}{4}\right)} = 2.824\cdots.$$

Proof: Since the function $h(t) = t/2\sqrt{3+t^2}$ is increasing on $[0, 1]$, we obtain

$$c(\Omega) \ge \frac{R(\Omega)}{2\sqrt{3+R(\Omega)^2}} = \frac{R(\Delta)}{2\sqrt{3+R(\Delta)^2}}$$

$$\ge \frac{\tanh(\frac{\pi c(\Delta)}{2})}{2\sqrt{3+\tanh^2\left(\frac{\pi c(\Delta)}{2}\right)}} = \frac{c(\Delta)}{2c(\Delta)\sqrt{1+3\coth^2\left(\frac{\pi c(\Delta)}{2}\right)}}.$$

Because $k(t) = 2t\sqrt{1+3\coth^2(\pi t/2)}$ is an increasing function of t and, by Theorem 4, $t = c(\Omega) \in [0, 1/2]$, we obtain $c(\Omega) \ge c(\Delta)/k(1/2)$. This establishes the lower bound; the upper bound follows by symmetry. ∎

Remark: Notice that the upper bound for $c(\Omega)/c(\Delta)$ obtained in the preceding theorem corresponds to the value $t = c(\Omega) = 1/2$. But by Theorem 4, if $c(\Omega) = 1/2$, then Ω is convex and so Ω and Δ are both simply connected. Then $c(\Omega)/c(\Delta) \le 2$. This supports our conjecture that the upper bound for $c(\Omega)/c(\Delta)$ should be 2 for all pairs of conformally equivalent uniformly perfect domains. Note that in the Corollaries to Theorems 1, 2 and 3 the extremal upper bound on the quotient of the various domain constants was attained for a pair of simply connected regions. This is further evidence for our conjecture.

THEOREM 6: *Let Ω be a hyperbolic region in $\mathbb{C}$. Then Ω is uniformly perfect if and only if one of the following equivalent conditions holds:*

(i) $\eta(\Omega) < \infty$;

(ii) $\beta(\Omega) < \infty$;

(iii) $r(\Omega) > 0$;

(iv) $r_c(\Omega) > 0$.

Proof: This is an immediate consequence of Theorems 1, 2, 3 and 5. ∎

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